Five Series Equations involving Heat Polynomials of second kind

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***Abstract* -In this paper, it is shown that five series equations involving heat polynomials of second kind can be solved by reducing them to simultaneous Fredholm integral equations of second kind .These equations are not considered earlier by [3],[4].**

***Keywords* - Integral equation, Series equation, Fourier series, Heat equations, Integral theorems**

I. INTRODUCTION

Dual,triple and quadruple series equations play an important role in finding the solution of mixed boundary value problems of elasticity, electrostatics and other fields of mathematical physics.Dual and triple equations involving orthogonal polynomials have been considered by many authors [1,3,4,5]. Cooke [2] devised a method for finding the solution of quadruple series equations involving Fourier- Bessel series and obtained the solution using operator theory. In this paper, we have considered five series equations involving heat polynomials which are extensions of dual, triple and quadruple series equations considered by authors[1,3,4,5].

II. FIVE SERIES EQUATIONS

We consider here the following sets of the five series equations of second kind:

Five series equations of the second kind are as follows:

$$\sum_{n=0}^{\infty}\frac{t^{-n}l^{n}B_n}{\left(\boldsymbol{\nu}+n+\frac{1}{2}+\rho\right)}P_{n+\rho,\nu}(x,-t)=$$

$$\begin{cases} g_1(x,t) & ,0\ \leq x<a \\ g_3(x,t) & ,b<x<c \\ g_5(x,t) & ,d<x<\infty \end{cases} \quad \textbf{(1)}$$

$$\sum_{n=0}^{\infty}\frac{B_n}{\left(\mu+n+\frac{1}{2}+\rho\right)}P_{n+\rho,\sigma}(x,t)$$

$$=\begin{cases} g_2(x,t) & ,a<x<b \\ g_4(x,t) & ,c<x<d \end{cases} \quad \textbf{(2)}$$

Where$g_i(x,t)$are unknown functions for (i=1,2,3,4,5).$P_{n,\nu}(x,-t)$isa heat polynomial andCoefficients $B_n$to be determined.

II. PRELIMINARY RESULTS

In the course of analysis, we shall use the following results

(i) The orthogonallyrelation for the heat

polynomials

$$\int_0^\infty W_{m,\nu}(x,t)P_{n,\nu}(x,-t)d\boldsymbol{\Omega}(\boldsymbol{x}) = \frac{\boldsymbol{\delta_{mn}}}{\boldsymbol{K_n}} \qquad \textbf{(3)}$$

whereδ_{mn} is the Kronecker delta,

$$\mathrm{d}\Omega\,(\mathrm{x}) = 2^{\frac{1}{2}-\boldsymbol{\nu}}\,[\Gamma\left(\upsilon+\frac{1}{2}\right)]^{-1}x^{2\upsilon}dx \qquad \textbf{(4)}$$

and ,$K_n = \frac{\Gamma\left[\upsilon+\frac{1}{2}\right]}{2^{4n}n!\Gamma\left[\upsilon+\frac{1}{2}+n\right]}$**(5)**

(ii) The series ,

$$S(x,\xi,t) = 2^{\frac{1}{2}-\sigma}\sum_{n=0}^{\infty}\frac{\left(\frac{l}{t}\right)^n\Gamma\left(\mu+\frac{1}{2}+n+\rho\right)P_{n+\rho,\upsilon}(x,-t)W_{n+\rho,\sigma}(\xi,t)}{2^{4(n+\rho)}(n+\rho)!\,\Gamma\left(\sigma+\frac{1}{2}+n+\rho\right)\Gamma\left(\upsilon+\frac{1}{2}+n+\rho\right)}\textbf{(6)}$$

$$S(x,\xi,t) = \frac{x^{1-2\upsilon}\xi^{1-2\sigma}e^{-\frac{\xi^2}{4t}}}{\Gamma m\Gamma(\upsilon-\sigma\ +m)}a_n{}^*$$

$$\int_0^\omega \eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{\upsilon-\sigma+m-1}dy$$

Where,

$$a_n{}^* = \frac{l^n t^{-(m+n)}2^{2(1-m)}\Gamma\left(\mu+\frac{1}{2}+n+\rho\right)}{\Gamma\left(\sigma-m+\frac{1}{2}+n+\rho\right)},$$

$(\upsilon-\sigma+m>0)$**(8)**

$\eta(y) = y^{2(\sigma-m)}e^{y^2/4t}$**(9)**

and, $\omega = \min(\xi,x)$

(iii)Ifh(y) is strictly monotonically increasing and differentiable function in (a,b) and h(y) ≠0 in this interval, then the solutions to the Abel integral equations.

$f(x) = \int_a^x \frac{\Phi(y)}{\{h(x)-h(y)\}^\alpha}dy\,, 0<\alpha<1$ **(10)**

and

$f(x) = \int_x^b \frac{\Phi(y)}{\{h(y)-h(x)\}^\alpha}dy\,, 0<\alpha<1$ **(11)**

are given by ,

$$\Phi(y) = \frac{\sin(\alpha\pi)}{\pi}\frac{d}{dy}\int_a^y\frac{h'(x)F(x)}{\{h(y)-h(x)\}^{1-\alpha}}dx$$

(12)

and

$$\Phi(y) = -\frac{\sin(\alpha\pi)}{\pi}\frac{d}{dy}\int_y^b \frac{h'(x)F(x)}{\{h(x)-h(y)\}^{1-\alpha}}dx \qquad \textbf{(13)}$$

respectively.

II. THE SOLUTION OF FIVE SERIES EQUATIONS OF THE SECOND KIND

Let us assume that ,

$$\sum_{n=0}^{\infty}\frac{B_n P_{n+\rho,\sigma}(x,-t)}{\Gamma\left(\mu+\frac{1}{2}+n+\rho\right)} = \begin{cases}\Psi_1(x,t), & 0<x<a\\ \Psi_2(x,t), & b<x<c\\ \Psi_3(x,t), & d<x<\infty\end{cases}$$

(14)

where $\Psi_1(x,t) and \Psi_2(x,t)$ are unknown functions.

Using orthogonally relation(3), we get B_n from equations(1),(2)

$$B_n = \frac{\Gamma\left(\sigma+\frac{1}{2}\right)\Gamma\left(\mu+\frac{1}{2}+n+\rho\right)}{2^{4(n+\rho)}(n+\rho)!}\left\{\int_0^a \Psi_1(x,t)+\int_a^b g_2(x,t)+\int_b^c \Psi_2(x,t)+\int_c^d g_4(x,t)+\int_d^{\infty}\Psi_3(x,t)\right\}W_{n+\rho,\sigma}(x,t)d\Omega(x) \textbf{(15)}$$

Now substituting the value of B_n from (15) in equation (2), we get,

$$\int_0^a \Psi_1(\xi,t)S(x,\xi,t)d\Omega(\xi)+\int_a^b g_2(\xi,t)S(x,\xi,t)d\Omega(\xi)+\int_b^c \Psi_2(\xi,t)S(x,\xi,t)d\Omega(\xi)+\int_c^d g_4(\xi,t)S(x,\xi,t)d\Omega(\xi)$$
$$+\int_d^{\infty}\Psi_3(\xi,t)S(x,\xi,t)d\Omega(\xi)$$
$$=\frac{2^{\frac{1}{2}-\sigma}}{\Gamma(\sigma+\frac{1}{2})}\begin{cases}g_1(x,t), & 0<x<a\\ g_3(x,t), & b<x<c\\ g_5(x,t), & d<x<\infty\end{cases} \textbf{(16)}$$

where S(x,t) is defined by equation (6).

Now starting with equation (16), which can be written as

$$\int_0^a \Psi_1(\xi,t)S(x,\xi,t)d\Omega(\xi)+\int_b^c \Psi_2(\xi,t)S(x,\xi,t)d\Omega(\xi)+\int_d^{\infty}\Psi_3(\xi,t)S(x,\xi,t)d\Omega(\xi) = \frac{2^{\frac{1}{2}-\sigma}}{\Gamma\left(\sigma+\frac{1}{2}\right)}\cdot\begin{cases}G_1(x,t), & 0<x<a\\ G_2(x,t), & b<x<c\\ G_3(x,t), & d<x<\infty\end{cases} \textbf{(17)}$$

where,

$$G_1(x,t) = g_1(x,t)-\int_a^b \xi^{2\sigma}g_2(\xi,t)S(x,\xi,t)d(\xi)-\int_c^d \xi^{2\sigma}g_4(\xi,t)S(x,\xi,t)d(\xi) \textbf{(18)}$$

with the help of equations (4), (7), we get $\int_0^x \Psi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi$

$$.\int_0^{\xi} \eta(y)(\xi^2 - y^2)^{m-1}(x^2 - y^2)^{\upsilon-\sigma+m-1} dy + \int_x^{a} \Psi_1\,(\xi,t)\xi e^{-\frac{\xi^2}{4t}} d\xi$$

$$.\int_0^{x} \eta(y)(\xi^2 - y^2)^{m-1}(x^2 - y^2)^{\upsilon-\sigma+m-1} dy$$

$$= \frac{\Gamma m \Gamma(\upsilon - \sigma + m)}{a_n * x^{1-2\upsilon}} G_1(x,t) - \int_b^{c} \Psi_2\,(\xi,t)\xi e^{-\frac{\xi^2}{4t}} d\xi . \int_0^{x} \eta(y)(\xi^2 - y^2)^{m-1}(x^2 - y^2)^{\upsilon-\sigma+m-1} dy$$

$$- \int_d^{\infty} \Psi_3\,(\xi,t)\xi e^{-\frac{\xi^2}{4t}} d\xi$$

$$.\int_0^x \eta(y)(\xi^2 - y^2)^{m-1}(x^2 - y^2)^{\upsilon-\sigma+m-1} dy \quad \mathbf{(19)}$$

Inverting the order of integration, we obtain

$$\int_0^{x} \eta(y)(x^2 - y^2)^{\upsilon-\sigma+m-1} dy$$

$$.\int_a^{x} \Psi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2 - y^2)^{m-1} d\xi$$

$$+ \int_0^{x} \eta(y)(x^2 - y^2)^{\upsilon-\sigma+m-1} dy$$

$$.\int_x^{a} \Psi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2 - y^2)^{m-1} d\xi$$

$$= \frac{\Gamma m \Gamma(\upsilon - \sigma + m)}{a_n * x^{1-2\upsilon}} G_1(x,t)$$

$$- \int_0^{x} \eta(y)(x^2 - y^2)^{\upsilon-\sigma+m-1} dy$$

$$.\int_b^{c} \Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2 - y^2)^{m-1} d\xi$$

$$- \int_0^{x} \eta(y)(x^2 - y^2)^{\upsilon-\sigma+m-1} dy$$

$$.\int_d^{\infty} \Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2 - y^2)^{m-1} d\xi \quad \mathbf{(20)}$$

$$\int_0^x \frac{\eta(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy\int_y^a \frac{\Psi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi = \frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a_n * x^{1-2\upsilon}}G_1(x,t)$$
$$-\int_0^x \frac{\eta(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy\int_b^c \frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi$$
$$-\int_0^x \frac{\eta(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy\int_d^\infty \frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi \quad (\mathbf{21})$$

If we assume $\int_y^a \frac{\Psi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi = \overline{\Psi}_1(y)$ **(22)**

$\int_y^c \frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi = \overline{\Psi}_2(y)$**(23)**

$\int_y^\infty \frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi = \overline{\Psi}_3(y)$**(24)**

Then equation (21) can be rewritten as below:

$$\int_0^x \frac{\eta(y)\overline{\Psi}_1(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy = \frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a_n * x^{1-2\upsilon}}G_1(x,t)$$

$$-\int_0^x \frac{\eta(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy\int_b^c \frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi$$

$$-\int_0^x \frac{\eta(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy\int_d^\infty \frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi\ ,$$

This is an Abel type integral equation and its solution, with the help of equation (12)is given by,

$$\eta(y)\overline{\Psi}_1(y) = \frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}\frac{d}{dy}\int_0^y \frac{2x}{(y^2-x^2)^{\upsilon+\sigma-m}}$$

$$\cdot\left[\frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a_n * x^{1-2\upsilon}}G_1(x,t) - \int_0^x \frac{\eta(z)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dz\int_b^c \frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right.$$
$$\left.-\int_o^x \frac{\eta(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dz\int_d^\infty \frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right]dx$$

Or,

$$\eta(y)\overline{\Psi}_1(y) = G'_1(y,t) - \frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}\left[\int_0^a \eta(z)dz\frac{d}{dy}\int_z^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}\int_b^c \frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi + \left[\int_0^x \eta(z)dz\frac{d}{dy}\int_z^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}\int_d^\infty \frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right]\right.$$

Where **(25)**

$$G'_1(y,t) = \frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}\;\frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a_n{}^*}\frac{d}{dy}\int_0^y \frac{2x^{2\upsilon}G_1(x,t)dx}{(y^2-x^2)^{\upsilon+\sigma-m}}\quad\textbf{(26)}$$

Inverting the order of integration in the II integral of R.H.S. in equation (24),we get,

$$\eta(y)\overline{\Psi}_1(y) = G'_1(y,t) - \frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}.\left[\int_0^y \eta(z)dz\frac{d}{dy}\int_z^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}.\int_b^c \frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi + \int_0^y \eta(z)dz\frac{d}{dy}\int_z^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}\int_d^\infty \frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right]$$

(27) It can be easily proved that,

$$\frac{d}{dy}\int_a^y \frac{2xdx}{(y^2-x^2)^{\upsilon-\sigma+m}(x^2-z^2)^{1-\upsilon+\sigma-m}} = \frac{(a^2-z^2)^{\upsilon-\sigma+m}}{(y^2-z^2)(y^2-a^2)^{\upsilon-\sigma+m}}\quad\textbf{(28)}$$

and

$$\int_z^y \frac{2xdx}{(y^2-x^2)^{\upsilon-\sigma+m}(x^2-z^2)^{1-\upsilon+\sigma-m}} = \frac{\pi}{\sin(1-\upsilon+\sigma-m)\pi}\quad\textbf{(29)}$$

Now using the results (28) and (29) in equation (27),we obtain

$$\eta(y)\overline{\Psi}_1(y) = G'_1(y,t)-.\left[\frac{d}{dy}\int_0^y \eta(z)dz\int_b^c \frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi + \frac{d}{dy}\int_0^y \eta(z)dz\int_d^\infty \frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right]\quad\textbf{(30)}$$

Equations (22) and (23) are also Abel type integral equations. Therefore the solution of these equations are given by

$$\Psi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}} = \frac{-\sin(1-m)\pi}{\pi}\frac{d}{d\xi}\int_\xi^a \frac{2y\overline{\Psi}_1(y)dy}{(y^2-\xi^2)^m}\quad\textbf{(31)}$$

$$\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}} = -\frac{\sin(1-m)\pi}{\pi}\frac{d}{d\xi}\int_\xi^c \frac{2y\overline{\Psi}_2(y)dy}{(y^2-\xi^2)^m}\quad\textbf{(32)}$$

$$\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}} = -\frac{\sin(1-m)\pi}{\pi}\frac{d}{d\xi}\int_\xi^\infty \frac{2y\overline{\Psi}_3(y)dy}{(y^2-\xi^2)^m}\quad\textbf{(33)}$$

Therefore,

$$\int_0^a \frac{\Psi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi = \frac{\sin(1-m)\pi}{\pi(a^2-z^2)^{-m}}\int_0^a \frac{2x\overline{\Psi}_1(x)dx}{x^{2m}(x^2-z^2)}\quad\textbf{(34)}\quad \int_b^c \frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi = \frac{\sin(1-m)\pi}{\pi(b^2-z^2)^{-m}}\int_b^c \frac{2x\overline{\Psi}_2(x)dx}{(x^2-b^2)^m(x^2-z^2)}$$

(35)

$$\int_{d}^{\infty} \frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi = \frac{\sin(1-m)\pi}{\pi(d^2-z^2)^{-m}} \int_{d}^{\infty} \frac{2x\overline{\Psi}_3(x)dx}{(x^2-d^2)^m(x^2-z^2)}$$

(36)

Substituting the values from equations (33)and (34) in equation (30), and changing the order of integration we obtain

$$\eta(y)\overline{\Psi}_1(y) = G'_1(y,t) - \int_{b}^{c} \frac{2x\overline{\Psi}_2(x)dx}{(x^2-b^2)^m}.$$

$$\left\{\frac{\sin(1-m)\pi}{\pi}\frac{d}{dy}\int_0^y \frac{\eta(z)(b^2-z^2)^m dz}{(x^2-z^2)}\right\} - \int_d^\infty \frac{2x\overline{\Psi}_3(x)dx}{(x^2-d^2)^m}$$

$$\left\{\frac{\sin(1-m)\pi}{\pi}\frac{d}{dy}\int_0^y \frac{\eta(z)(d^2-z^2)^m dz}{(x^2-z^2)}\right\}$$

This equation takes the form

$\eta(y)\overline{\Psi}_1(y) = F'_1(y,t) - \int_a^b \overline{\Psi}_1(x)K_1(x,y)\,dx - \int_c^d \overline{\Psi}_2(x)K_2(x,y)\,dx\,,\ a < x < b$ **(37)**

where, $K_1(x,y) = \frac{\sin(1-m)\pi}{\pi}.$

$\frac{2x}{(x^2-b^2)^m}\frac{d}{dy}\int_0^y \frac{\eta(z)(b^2-z^2)^m}{(x^2-z^2)}\,dz$ **(38)**

$$K_2(x,y) = \frac{\sin(1-m)\pi}{\pi}\frac{2x}{(x^2-d^2)^m}\frac{d}{dy}\int_0^y \frac{\eta(z)(d^2-z^2)^m}{(x^2-z^2)}\,dz$$

(39)

Now starting with equation (16)

$$\int_0^a \Psi_1(\xi,t)S_\xi(x,\xi,t)d\Omega(\xi) + \int_b^x \Psi_2(\xi,t)S_\xi(x,\xi,t)d\Omega(\xi) + \int_x^c \Psi_2(\xi,t)S_x(x,\xi,t)d\Omega(\xi) + \int_d^\infty \Psi_3(\xi,t)S_x(x,\xi,t)d\Omega(\xi)$$

$= \frac{2^{\frac{1}{2}-\sigma}}{\Gamma\left(\sigma+\frac{1}{2}\right)}G_2(x,t)\quad ,b < x < c$ **(40)**with the help of equations (4) and (7),we get

$$\int_b^x \Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi$$

$$\int_0^{\xi} \eta(y)(\xi^2 - y^2)^{m-1}\,(x^2 - y^2)^{\upsilon-\sigma+m-1} dy + \int_x^{c} \Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}} d\xi$$

$$\int_0^{x} \eta(y)(\xi^2 - y^2)^{m-1}\,(x^2 - y^2)^{\upsilon-\sigma+m-1} dy$$

$$= \frac{\Gamma m \Gamma(\upsilon - \sigma + m)}{a * x^{1-2\nu}} G_2\,(x,t) - \int_0^{a} \Psi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}} d\xi$$

$.\int_0^{\xi} \eta(y)(\xi^2 - y^2)^{m-1}\,(x^2 - y^2)^{\upsilon-\sigma+m-1} dy$

$$- \int_d^{\infty} \Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}} d\xi$$

$\int_0^{x} \eta(y)(\xi^2 - y^2)^{m-1}\,(x^2 - y^2)^{\upsilon-\sigma+m-1} dy$ **(41)**

Now inverting the order of integration, we obtain,

$$\int_0^{b} \eta(y)(x^2 - y^2)^{\upsilon-\sigma+m-1} dy$$

$$.\int_b^{x} \Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}\,(\xi^2 - y^2)^{m-1} d\xi$$

$$+ \int_b^{x} \eta(y)(x^2 - y^2)^{\upsilon-\sigma+m-1} dy$$

$$.\int_y^{x} \Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}\,(\xi^2 - y^2)^{m-1} d\xi$$

$$+ \int_0^{x} \eta(y)(x^2 - y^2)^{\upsilon-\sigma+m-1} dy$$

$$.\int_x^{c} \Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}\,(\xi^2 - y^2)^{m-1} d\xi$$

$$= \frac{\Gamma m \Gamma(\upsilon - \sigma + m)}{a * x^{1-2\nu}} G_2\,(x,t) -$$

$$\int_0^{a} \eta(y)(x^2 - y^2)^{\upsilon-\sigma+m-1} dy$$

$$.\int_{y}^{a} \Psi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi$$

$$-\int_{0}^{x} \eta(y)(x^2-y^2)^{\upsilon-\sigma+m-1}dy$$

$.\int_{d}^{\infty} \Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi$**(42)**

Or

$$\int_{b}^{x} \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} \int_{y}^{x} \frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi \quad = \frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a*x^{1-2\nu}}G_2(x,t)$$

$$-\int_{0}^{a} \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} \int_{y}^{x} \frac{\Psi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi - \int_{0}^{b} \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} \int_{b}^{c} \frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi$$

$$-\int_{0}^{x} \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} \int_{d}^{\infty} \frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi \qquad \mathbf{(43)}$$

Using equations (22) and (23), then equation (43) can be rewritten as below:

$$\int_{b}^{x} \frac{\eta(y)\overline{\Psi}_2(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} = \frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a*x^{1-2\nu}}G_2(x,t) -$$

$$\int_{0}^{a} \frac{\eta(y)\overline{\Psi}_1(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} - \int_{b}^{c} \frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi$$

$$\int_{0}^{b} \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} - \int_{0}^{x} \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}}$$

$$.\int_{d}^{\infty} \frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi \qquad , c<x<d$$

This is an Abel type integral equation and its solution,with the help of equation (12) is given by

$$\eta(y)\overline{\Psi}_2(y) = \frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}\frac{d}{dy}\int_{b}^{y} \frac{2x}{(y^2-x^2)^{\upsilon-\sigma+m}}.$$

$$\left[\frac{\Gamma m\Gamma(v-\sigma+m)}{a*x^{1-2v}}G_2(x,t)-\int_0^a\frac{\eta(z)\overline{\Psi}_1(z)dz}{(x^2-z^2)^{1-v+\sigma-m}}-\int_b^c\frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi.\int_0^b\frac{\eta(z)dz}{(x^2-z^2)^{1-v+\sigma-m}}-\int_0^x\frac{\eta(z)dz}{(x^2-z^2)^{1-v+\sigma-m}}\int_d^\infty\frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right]dx$$

Or,

$$\eta(y)\overline{\Psi}_2(y)=G'_2(y,t)-\frac{\sin(1-v+\sigma-m)\pi}{\pi}\left[\int_0^a\eta(z)dz.\frac{d}{dy}\int_b^y\frac{2x\overline{\Psi}_1(z)\,dx}{(y^2-x^2)^{v-\sigma+m}(x^2-z^2)^{1-v-\sigma+m}}\right.$$

$$+\int_0^b\eta(z)dz.\frac{d}{dy}\int_b^y\frac{2xdx}{(y^2-x^2)^{v-\sigma+m}(x^2-z^2)^{1-v-\sigma+m}}\int_b^c\frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi+$$

$$\int_0^b\eta(z)dz.\frac{d}{dy}\int_b^y\frac{2xdx}{(y^2-x^2)^{v-\sigma+m}(x^2-z^2)^{1-v-\sigma+m}}.\int_d^\infty\frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi+$$

$$\left.\frac{d}{dy}\int_b^y\eta(z)dz\int_z^y\frac{2xdx}{(y^2-x^2)^{v-\sigma+m}(x^2-z^2)^{1-v-\sigma+m}}.\int_d^\infty\frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right] \tag{44}$$

where,

$$G'_2(y,t)=\frac{\sin(1-v+\sigma-m)\pi}{\pi}.\frac{\Gamma m\Gamma(v-\sigma+m)}{a*}.\frac{d}{dy}\int_b^y\frac{2x^{2v}G_2(x,t)}{(y^2-x^2)^{v-\sigma+m}}dx$$

Using the equations (28) and (29) in equation (44),we obtain

$$\eta(y)\overline{\Psi}_2(y)=G'_2(y,t)-\frac{\sin(1-v+\sigma-m)\pi}{\pi}\left[\int_0^a\frac{\eta(z)(b^2-z^2)^{v-\sigma+m}dz}{(y^2-b^2)^{v-\sigma+m}(y^2-z^2)}\right.$$

$$.\int_z^a\frac{\Psi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi+\int_0^b\frac{\eta(z)(b^2-z^2)^{v-\sigma+m}dz}{(y^2-b^2)^{v-\sigma+m}(y^2-z^2)}.\int_b^c\frac{\Psi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi$$

$$\left.+\int_0^b\frac{\eta(z)(b^2-z^2)^{v-\sigma+m}dz}{(y^2-b^2)^{v-\sigma+m}(y^2-z^2)}.\int_d^\infty\frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi+\frac{\pi}{\sin(1-v+\sigma-m)\pi}\frac{d}{dy}\int_b^y\eta(z)dz\int_d^\infty\frac{\Psi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right] \tag{46}$$

Making use of equations (31) to (36) in equation (46), we obtain

$$\eta(y)\overline{\Psi}_2(y)=G'_2(y,t)-\frac{\sin(1-v+\sigma-m)\pi\sin(1-m)\pi}{\pi^2}.\left[\int_0^b\frac{\eta(z)(b^2-z^2)^{v-\sigma+2m}dz}{(y^2-b^2)^{v-\sigma+m}(y^2-z^2)}.\int_b^c\frac{2x\overline{\Psi}_2(x)dx}{(x^2-b^2)^m(x^2-z^2)}\right.$$

$$-\int_0^a\frac{\eta(z)(b^2-z^2)^{v-\sigma+m}dz}{(y^2-b^2)^{v-\sigma+m}(y^2-z^2)}.\int_z^a\frac{1}{(\xi^2-z^2)^{1-m}}\left\{\frac{d}{d\xi}\int_\xi^a\frac{2x\overline{\Psi}_1(x)dx}{(x^2-\xi^2)^m}\right\}+\int_0^b\frac{\eta(z)(b^2-z^2)^{v-\sigma+m}dz}{(y^2-b^2)^{v-\sigma+m}(y^2-z^2)(d^2-z^2)^{-m}}.$$

$$\int_d^\infty\frac{2x\overline{\Psi}_3(x)dx}{(x^2-d^2)^m(x^2-z^2)}+\frac{\pi}{\sin(1-v+\sigma-m)\pi}$$

$$\left..\frac{d}{dy}\int_b^y\frac{\eta(z)dz}{(d^2-z^2)^{-m}}\int_d^\infty\frac{2x\overline{\Psi}_3(x)dx}{(x^2-d^2)^m(x^2-z^2)}\right] \tag{47}$$

Changing the order of integration and using Leibnitz theorem, we find

$$\eta(y)\bar{\Psi}_2(y) = G'_2(y,t) - \frac{\sin(1-\upsilon+\sigma-m)\pi\sin(1-m)\pi}{\pi^2}$$

$$\left[\int_b^c \frac{2x\bar{\Psi}_2(x)dx}{(x^2-b^2)^m(y^2-b^2)^{\upsilon-\sigma+m}}\int_0^b \frac{\eta(z)(b^2-z^2)^{\upsilon-\sigma+2m}dz}{(y^2-z^2)(x^2-z^2)} -\right.$$

$$.\frac{1}{m^{-1}}\int_0^a \frac{\eta(z)(b^2-z^2)^{\upsilon-\sigma+m}dz}{(y^2-b^2)^{\upsilon-\sigma+m}(y^2-z^2)}.\int_z^a \frac{d\xi}{(\xi^2-z^2)^{1-m}}$$

$$.\int_\xi^a \frac{2\xi\ 2x\bar{\Psi}_1(x)dx}{(x^2-\xi^2)^{1+m}} + \int_d^\infty \frac{2x\bar{\Psi}_3(x)dx}{(x^2-d^2)^m(y^2-b^2)^{\upsilon-\sigma+m}}.$$

$$\left.\int_0^b \frac{\eta(z)(b^2-z^2)^{\upsilon-\sigma+m}(d^2-z^2)^m dz}{(y^2-z^2)(x^2-z^2)}\right]$$

The above equation takes the form,

$$\eta(y)\bar{\emptyset}_2(y) = F'_2(y,t) - \int_a^b \bar{\emptyset}_1(x)S(x,y)dx -$$

$$\int_c^d \bar{\emptyset}_2(x)R(x,y)dx$$

$$c < x < d \quad \textbf{(46)}$$

where,

$$\mathrm{S(x,y)} = \frac{\sin(1-\upsilon+\sigma-m)\pi\sin(1-m)\pi}{\pi^2(y^2-c^2)^{\upsilon-\sigma+m}}\left[\frac{2x}{(x^2-a^2)^m}\int_0^a \frac{\eta(z)(c^2-z^2)^{\upsilon-\sigma+m}(a^2-z^2)^m dz}{(y^2-z^2)(x^2-z^2)}\right.$$

$$\left.-\frac{1}{m^{-1}}\int_a^x \frac{2\xi d\xi}{(x^2-\xi^2)^{1+m}}\int_a^\xi \frac{\eta(z)\left(c^2-z^2\right)^{\upsilon-\sigma+m}dz}{(\xi^2-z^2)^{1-m}(y^2-z^2)}\right] \textbf{(47)}$$

And

$$\mathrm{R(x,y)} = \frac{\sin(1-\upsilon+\sigma-m)\pi\ \mathrm{si}\ \ (1-m)\pi}{\pi^2(y^2-c^2)^{\upsilon-\sigma+m}}\frac{2x}{(x^2-c^2)^m}\int_0^c \frac{\eta(z)\left(c^2-z^2\right)^{\upsilon-\sigma+2m}dz}{(y^2-z^2)(x^2-z^2)} \textbf{(48)}$$

Equations(34) and (46) are simultaneous Fredholm integral equations of the second kind.With the help of these equations we can calculate the values of $\bar{\emptyset}_1(y)$and $\bar{\emptyset}_2(y)$**.** Thenusing equations (30) and (31),we can calculate the values of $\emptyset_1(\xi,t)$and$\emptyset_2(\xi,t)$.After these calculations A_n can be determined by equation (15).

II. Conclusions

Whend → ∞ , then above equations reduce to quadruple series equations and our solution agrees with that obtained in [4].When d → ∞and c → d ,the series reduces to triple series equations and solution obtained here agrees with[3].

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