A Method for Discretizing Continuous Time Controller Using Time Moments and Frequency Domain Approach

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Abstract: **In this paper, a novel method is introduced both using time moments and frequency domain approach. The method leads to linear algebraic equation in terms of unknown parameters of discretized controller. The frequency domain approach is performed using Pade technique for the digitized and continuous time systems. The time moments approach automatically computes the digital equivalent of an analog controller, which is fully implemented using MATLAB software. The time moments approach gives good performance at higher sampling period also.**

Keywords-Analog Controllers, Digital Controllers, Redesign

I. Introduction

For simulation studies as well as the design and implementation of computer controlled systems, discretization of continuous time controller is often required. Tabek used the prewarped bilinear transformation [2], while Kuo et al matched the time domain trajectories to arrive at the digital controller [3]. In Rattan [4], the digitized controller is found by minimizing an error between the continuous frequency response of the digital and analog systems in the bilinear transforms w domain.In this paper the parameters of the digitized controller are determined such that the closed loop frequency response of the digital systems approximates that of the continuous-time system in the Pade sense.

II. The Pade Approximation Technique And The Time Domain Matching

In general, a Pade approximant is a rational function $P_m(x)/Q_n(x)$, where $P_m(x)$ and $Q_n(x)$ are polynomials in x, of degrees m and n, respectively and is denoted by [m,n]. A rational function f(x), iff the power series expansion of [m,n] is identical with that of f(x) upto, and including terms of order x^{m+n}. Let the function to be approximated be defined by the power series;

$$f(x) = t_o + t_1 x + t_2 x^2 + \qquad (1)$$

and the Pade approximant be defined by:

$$\frac{P_m(x)}{Q_n(x)} = \sum_{i=0}^{m} P_i x^i \; / \; \sum_{i=0}^{n} q_i x^i \qquad (2)$$

Since the power series expansion of Eqn. (2) is to agree with Eqn. (1) as far as, and including, the term $x^{(m+n)}$, one may cross multiply after equating the right hand sides of Eqn. (1) and (2) to obtain.

$$\sum_{i=0}^{m} P_i x^i = \sum_{i=0}^{n} q_i x^i \; [t_0 + t_1 x + t_2 x^2 + \dots]$$

By equating on both sides, the coefficients of terms having equal powers in x, one obtain a set of (m+n+2) linear simultaneous algebraic equations in the unknown parameters of [m,n]. when [m,n] is obtained, we say:

$$\frac{P_m(x)}{Q_n(x)} \overset{P}{\Rightarrow} f(x) \qquad (3)$$

Where $\overset{P}{\Rightarrow}$ denotes equivalence in the Pade sense.

In the continuous-time, ie. S-domain, the coefficients of the power series expansion of G(s) about s=0 are called the time-moments. Equivalently, in the discrete time, i.e λdomain, the coefficients of the power series expansion of G(λ) about λ=1are called the time-moments. Proportional time-moments sequences may be obtained for G (λ) by using the bilinear transformation

$$\lambda=[1-(T/2)w]/\ [1+ (T/2)\ w] \qquad (4)$$

To obtain G(w), or a linear transformation $\lambda=1/(f+1)$,□to obtain G(f).

Where $\lambda=1/z$

The coefficient of the power series expansion of G(w) about w=0 or G(f)□about f=0, then give the proportional time moments of G(w) or G(f) in their respective domains. In the pade approximation technique[3], equivalence in the Pade sense vide Eqn. (III) implies that the initial few time moments of both the systems are identical and thus the low frequency characteristics of both the systems would be close. As most control systems are low pass, this concept of matching the initial few time moments has been utilized in the discretization method of this paper.

III. The Discretization Method

The block diagram of an existing continuous time controlled system is shown in figure 1, while the digitized system is shown in fig. 2 the overall transfer function of the continuous and digital system may be written as:

$Y(s) / R(s) = F(s) =$

$$= G_c(s)\ G_p(s) / [1+G_c(s)\ G_p(s)\ H(s)] \qquad (5)$$

$Y(\lambda) / R(\lambda) = G_d(\lambda) =$

$$= G_c(\lambda)\ G_h\ G_p(\lambda) / [1+G_c(\lambda)\ G_h\ G_p\ H(\lambda)] \qquad (6)$$

Where $G_c(s)$, $G_h(s)$, $G_p(s)$ and H(s) are the transfer function of the continuous controller, zero order hold, plant and feedback elements respectively, and $G_h\ G_p\ H(\lambda) = \lambda\{G_h(s)\ G_p(s)\}$

$$G_h\ G_p\ H(\lambda) = \lambda\{G_h(s)\ G_p(s)\ H(s)\} \qquad (7)$$

Given $G_c(s)$, the problem is to design $G_c(\lambda)$, in the proposed method, we find $G_c(\lambda)$ such that the frequency response of the open loop digital system $G_h\ G_p(\lambda)\ G_c(\lambda)$ approximates that of the equivalent open loop transfer function of the continuous time system in the Pade sense[10]. The open loop equivalent transfer function $M_q(s)$ of the system in fig. (1) is given by

$$M_q(s) = F(s)/[1- F(s)\ H(s)] \qquad (8)$$

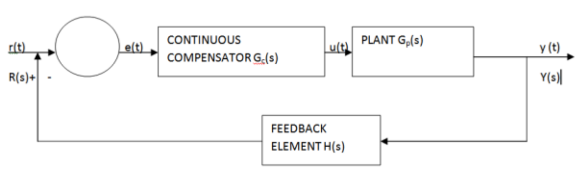

Fig.1 Existing continuous-data control system

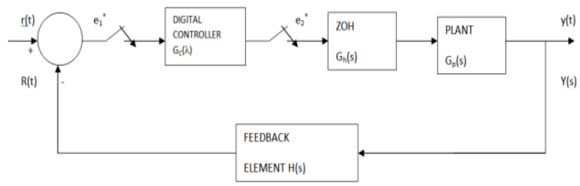

Fig.2 redesigned digital control system

The discretization procedure may be given in the following steps:

Step I: Given the system of Fig. 1, the system of Fig. 2 must follow the w domain model

$$M(w) = F(s)\ |_{s=w} \qquad (9)$$

Step II: The desired equivalent open loop model is

$$M_q(w) = M(w)/\ [1-M(w)\ H(w)] \qquad (10)$$

Step III: The digital controller in the w domain is specified as

$$G_c(w) = \sum_{i=0}^{m} a_i w^i / \sum_{i=0}^{n} b_i w^i \qquad (11)$$

Where $m \leq n$. This controller has (m+n+2) unknown coefficients.

Step IV: The open lop digital system of i.e. $G_c(\lambda)\ G_h\ G_p(\lambda)$ is transformed to w domain by using

$$\lambda=[1-(T/2)w]/\ [1+ (T/2)\ w] \qquad (12)$$

Step V: The design philosophy of this method is to equate the closed loop continuous (fig. 1) and digitized (fig. 2) system in the Pade sense (Eqn. 3). This is done by equating the equivalent open loop function $M_q(w)$ and $G_c(w)\ G_p(w)$ from step II and IV above.

$$G_c(w)\ G_p(w) = M_q(w) \qquad (13)$$

$$\text{or, } G_c(w) \sum_{i=0}^{\infty} g_i w^i = \sum_{i=0}^{\infty} m_i w^i \qquad (14)$$

Where $G_p(w)$, $M_q(w)$ are assumed to have formal power series expansions about w = 0

$$G_p(w) = \sum_{i=0}^{\infty} g_i w^i \qquad (15)$$

$$M_q(w) = \sum_{i=0}^{\infty} m_i w^i \qquad (16)$$

For Eq.(13) to be valid in the Pade sense, assuming b_o=1; the (m+n+1) unknown parameters of $G_c(w)$ can be determined from the following matrix equation.

$$\begin{bmatrix} g_0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ g_1 & g_0 & 0 & -m_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ g_m & g_{m-1} & g_0 & -m_{m-1} & -m_{m-2} & \cdots & -m_{m-n} \\ g_{m+1} & g_m & g_1 & -m_m & -m_{m-1} & \cdots & -m_{m-n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ g_{m+n} & g_{m+n-1} & g_n & -m_{m+n-1} & -m_{m+n-2} & \cdots & -m_n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$= [m_0 \quad m_1 \quad \cdot \quad \cdot \quad m_m \quad m_{m+1} \quad \cdot \quad \cdot \quad m_{m+n}] \qquad (17)$$

Eqn. (17) can be solved to determine the unknown coefficients of $G_c(w)$.

StepVI: The controller in the λdomain, $G_c(w)$ is then found by using the inverse bilinear transformation,[5].

$$w = \frac{2}{T}(1-\lambda)/(1+\lambda) \qquad (18)$$

putting the λ=1/z then

$$w = \frac{2}{T}(z-1)/(z+1) \qquad (19)$$

IV. ALGORITHM

The block diagram of an existing continuous data control system has been shown in fig.3 can be digitized by inserting a sampler of duration T at the error input and replacing the continuous controller $G_c(s)$ by a digital controller $D_c(z)$ and the zero order hold as shown in fig.4 respectively as given below,[7].

$$\frac{C(s)}{R(s)} = M(s)\,\frac{Gp(s).Gc(s)}{1+Gc(s).Gp(s)} \qquad (20)$$

$$\frac{C(z)}{R(z)} = G_d\,(z) = \frac{Dc(z).Gp(z)}{1+Dc(z).Gp(z)} \qquad (21)$$

Where $G_c(s)$ and $G_p(s)$ are the transfer function of continuous controller and plant respectively; and $D_c(z)$ and

$G_p(z)$ are the transfer function of digital controller and plant preceded by zero order hold (ZOH) respectively.

The design of the digital controller should be such that the steady state output of the digital control system follows the desired output of the continuous model for all sinusoidal input within the frequency range of the interest.

In the design method the parameters of the digitized controllers are determined such that the open loop digital system $G_p(z)$ $D_c(z)$ approximates in the Pade sense with the open loop transfer function of their continuous time system.

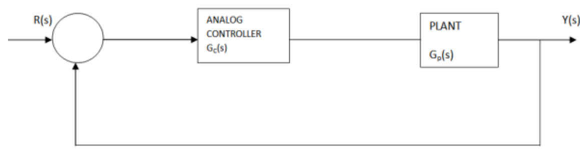

Fig.3continuous closed loop system

The open loop transfer function of the system shown in Fig.3 is given as:

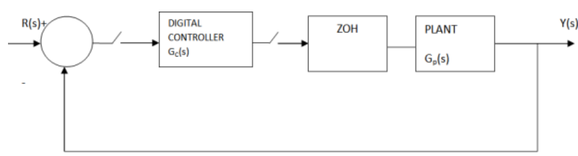

Fig.4 Discrete closed loop system

$$M_q(s) = \frac{M(s)}{1-M(s)}$$

The design procedure starts by converting the open loop transfer function in Fig.4 to w domain.The advantages of w transformation are that the sampling and data hold operations are modeled exactly in the w domain regardless of the sampling rate employs and the w variable is analogous to the 's' variable in the sense that all familiar frequency domain design concept procedure and interpretation are carried over directly.

Assuming $G_p(w)$ and $M_q(w)$ have formal power series expansion.

$$G_p(w) = \sum_{i=0}^{\infty} g_i w^i \qquad (22)$$

$$M_q(w) = \sum_{i=0}^{\infty} m_i w^i \qquad (23)$$

The design PID controller is designed by the T.F.

$$D(z) = K_p + \frac{K_I T(z+1)}{2(z-1)} + \frac{K_D(z-1)}{Tz} \qquad (24)$$

$$= \frac{C_0 z^2 + C_1 z + C_2}{z(z-1)}$$

Where,

$$C_0 = K_p + \frac{K_i T}{2} + \frac{K_D}{T}$$

$$C_1 = -K_p + \frac{K_i T}{2} - \frac{2K_D}{T}$$

$$C_2 = \frac{K_D}{T}$$

The unknown parameters of the digital controller are obtained by

$$D(w).G_p(p) = M_q(w) \qquad (25)$$

Where D(w) is obtained by substituting $z = \frac{1+T/2w}{1-T/2w}$

In D(z) and $\underline{\overset{P}{}}$ denotes equivalence in pade sense.

Let, $D(w) = \frac{d_0 w^2 + d_1 w + d_2}{(1+T/2w)Tw}$,then from (21), we get

$$\frac{d_0 w^2 + d_1 w + d_2}{(1+T/2w)Tw}(g_0 + g_1 w + g_2 w + .) = m_0 w^{-1} + m_1 w + m_2 w^1$$

Computing the like powers of w, we get

$d_2 g_0 = m_0 T$

$d_1 g_0 + d_2 g_1 = m_1 T + m_0 \frac{T^2}{2}$

$d_0 g_0 + d_1 g_1 + d_2 g_2 = m_2 T + m_1 \frac{T^2}{2}$

In matrix form,

$$\begin{bmatrix} g_0 & 0 & 0 \\ g_1 & g_0 & 0 \\ g_2 & g_1 & g_0 \end{bmatrix} \begin{bmatrix} d_2 \\ d_1 \\ d_0 \end{bmatrix} = \begin{bmatrix} m_0 T \\ m_1 T + \frac{m_0 T^2}{2} \\ m_2 T + m_1 \frac{T^2}{2} \end{bmatrix}$$

Which can be solved for d_2, d_1 and d_0? Thus controller D(w) is completely specified. Using the transformation

$w = \frac{2}{T}(z-1)/(z+1)$

the controller D(z) can be obtained. Finally using eq. (24), we can get the parameters of the digital PID controller.

EXAMPLE-1-Consider the example given by Rattan, where the transfer function of the continuous controller, plant and the feedback element are given by

$$G_C(s) = \frac{(1+0.416s)}{(1+0.139s)}$$

$$G_p(s) = \frac{10}{(s^2+s)}, \quad H(s) = 1$$

The continuous-time controller $G_c(s)$ is to be replaced by a digital controller $G_C(\lambda)$.Eq.(5) gives

$$F(s) = \frac{29.93 + 71.942}{s^3 + 8.194s^2 + 37.12s + 71.942} \qquad (26)$$

For a sampling period of T=0.15s, Eq.(3) gives

$G_H G_P(\lambda) = G_H G_P H(\lambda)$

$=(0.101858\lambda + □10708)/(0.8607\lambda^2 - 1.807\lambda + 1)$
(27)

Putting the $\lambda = [1-(T/2)w]/[1+(T/2)w]$ in Eq.(23)

$$G_p(w) = \frac{-0.29373E-4W^2 - 0.01527W + 0.20894}{W(0.0209W + 0.02089)} \qquad (28)$$

The power series expansion of Eq. (24) gives:

$$G_p(w) = 10w^{-1} - 10.75 + 10.76w- \qquad (29)$$

The reference model in the w domain is obtained from Eq. (26) as:

$M(w) = F(s)|_{s=w} = F(w)$

Using Eq.(6) the equivalent open loop model is,

$$M_q(w) = \frac{29.93w + 71.942}{w^3 + 8.194w^2 + 7.19w}$$

$$= 10w^{-1} - 7.24 + 6.86w- \qquad (30)$$

The w-domain controller is chosen as

$$G_C(W) = \frac{a_1 w + a_0}{b_1 w + b_0} \; ; \quad b_0 = 1$$

Substituting g_i 's and m_i 's from Eq.(29), (30) into (17), we get:

$$\begin{bmatrix} 10 & 0 & 0 \\ -10.75 & 10 & -10 \\ 10.76 & -10.75 & 7.24 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 10 \\ -7.24 \\ 6.86 \end{bmatrix} \quad (31)$$

Solving Eq. (31), we get:

$a_0=1$, $a_1=0.387$, $a_2=0.0364$

Thus, $G_c(w) = \frac{1+0.387w}{1+0.0364w}$

Using the inverse bilinear transformation of Eq.(18), we obtain

$G_c(\lambda)=4.02(1-0.6781\lambda)/(1+0.2941\lambda)$

putting the $\lambda=1/z$ then

$$G_C(z)=4.02\ \frac{z-0.6781}{z+0.2941}$$

Rattan' method gives the following controller,[4]:

$$G_{or}=3.436\frac{z-0.6377}{z+0.239}$$

A comparison of unit step response of the continuous and the digitally redesigned systems with the controllers $G_c(z)$ and $G_{or}(z)$ is shown in fig.5, while the corresponding frequency response plots are given in fig.6. The various figure of merit of the continuous and digital system are given in table

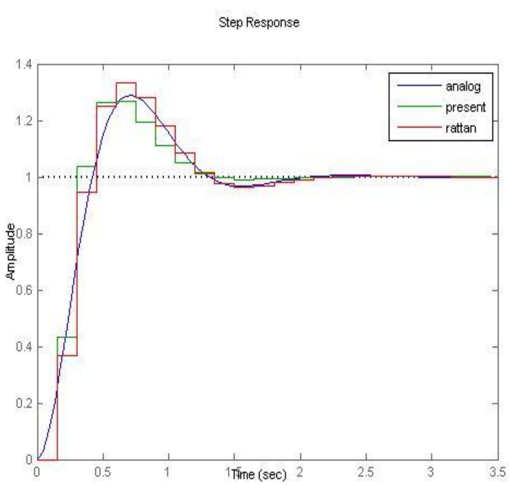

Fig .5 unit step response of the continuous model and the digital control system

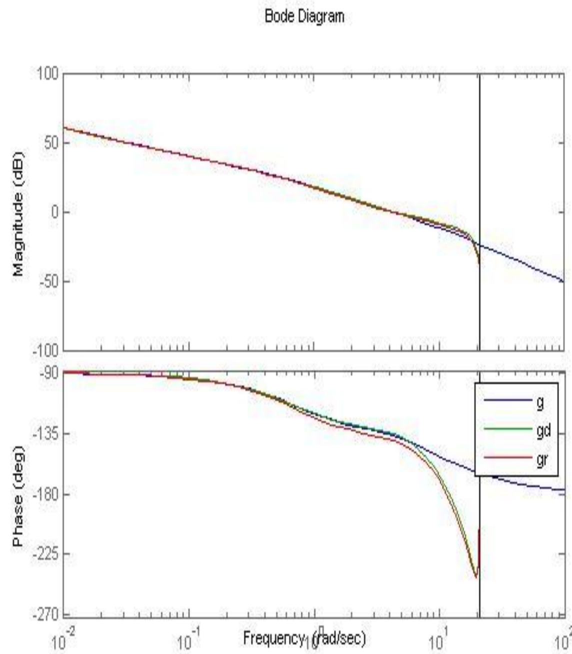

Fig.6 bode plot of the open loop transfer function

Where g – continuous system

g_{d-} present method

g_r-rattan method

TABLE I COMPARISON OF PERFORMANCE OF DIGITIZATION METHODS.

	Existing continuous system	Rattan	Proposed method
Gain margin (db)	∞	13.4	15.6
Phase margin (degrees)	41.38	32.12	33.19
Bandwidth (rads/sec.)	6.7	7.07	7.0
Peak Amplitude	1.29	1.33	1.28
Peak overshoot (%)	28.8	33.1	27.9
Rise Time (sec.)	0.287	0.245	0.271

V. CONCLUSION

In this paper is described for converting an existing continuous time controller into a digital one so that it gives similar performance. Digital redesign in the bilinear w-domain is carried out by using the idea of approximate model matching in the Pade sense. The method calls for the solution of set of linear algebraic equation only. The various responses for the example considered are compared with the method of Rattan. The method of Rattan calls for evaluation of integrals & is computationally more involved. In spite of its computational simplicity, this method compares favorably (see result) with that of Rattan where the frequency response deviation are minimized by a weighted least square complex curve fitting procedure.

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