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A Study of Relativistic Wave Equation of Electron

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***Abstract*—In this paper we have studied the relativistic wave equation of electron which helps in the study of papers [5] and [6] of Prof. Harish-Chandra (1923-1983), a Kanpur born distinguished mathematician of the world.**

***Keywords*—momentum; Hamiltonian; wave function; Schrödinger wave equation; free particle; linear operator;**

I. INTRODUCTION

For the non-relativistic consideration the energy, $E = E^{'} - mc^2$ for the free particle is just the kinetic energy, hence let us consider the function

$$\psi = e^{imc^2/\hbar}\varkappa$$

$$\psi = e^{\frac{i}{\hbar}(p_x x + p_y y + p_z z - Et)},$$

where

$$E = \tfrac{1}{2} m v^2 = \tfrac{1}{2m} m^2 v^2 = \tfrac{1}{2m} p^2.$$

In this case we get

$$i\hbar \frac{\partial \psi}{\partial t} = E\psi$$

$$-i\hbar \overline{\nabla} \psi = \bar{\mathrm{p}} \psi, \tag{1}$$

and

$$\nabla^2 \psi = -\frac{p^2}{\hbar^2}\psi = -\frac{2mE}{\hbar^2}\psi.$$

Thus we get the equation

$$\nabla^2 \psi + \frac{2mE}{\hbar^2}\psi = 0, \tag{2}$$

which is called *non-relativistic wave equation* for a free particle [10], [11].

From the relation (1) we have

$$-i\hbar \overline{\nabla} \psi = \bar{\mathrm{p}} \psi$$

$$-i\hbar \overline{\nabla} \equiv \bar{\mathrm{p}} \,,$$

which gives

$$\left.\begin{aligned} -i\hbar \frac{\partial}{\partial x} &\equiv p_x \\ -i\hbar \frac{\partial}{\partial y} &\equiv p_y \\ -i\hbar \frac{\partial}{\partial z} &\equiv p_z \end{aligned}\right\} \tag{3}$$

From the wave equation (2) may be written as

$$-\frac{\hbar^2}{2\mathrm{m}} \nabla^2 \psi = E\psi,$$

i.e.

$$\frac{1}{2m}\left[(-i\hbar)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right]\psi = E\psi$$

$$\frac{1}{2m}\left[\left(-i\hbar \frac{\partial}{\partial x}\right)^2 + \left(-i\hbar \frac{\partial}{\partial y}\right)^2 + \left(-i\hbar \frac{\partial}{\partial z}\right)^2\right]\psi = E\psi$$

$$\frac{1}{2m}\left[p_x^2 + p_y^2 + p_z^2\right]\psi = E\psi \tag{4}$$

$$H\psi = E\psi.$$

Here H is non-relativistic Hamiltonian for free motion of the particle i.e.

$$H = \frac{1}{2m}\left[p_x^2 + p_y^2 + p_z^2\right],$$

where p_x, p_y, p_z are differential operators defined by the equation (3).

If we now suppose that the particle is not free but external force applied on it is given by potential function V i.e. external force$= -grad\, V$, then in the equation (4) E will be replaced by $E - V$ and we get the wave equation in the form

$$\frac{1}{2m}\left[p_x^2 + p_y^2 + p_z^2\right]\psi = (E - V)\psi$$

$$\frac{1}{2m}\left[\left(-i\hbar \frac{\partial}{\partial x}\right)^2 + \left(-i\hbar \frac{\partial}{\partial y}\right)^2 + \left(-i\hbar \frac{\partial}{\partial z}\right)^2\right]\psi = (E - V)\psi$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi + \frac{2m}{\hbar^2}(E - V)\psi = 0$$

$$\nabla^2 \psi + \frac{2m}{\hbar^2}(E - V)\psi = 0, \tag{5}$$

where E is the total energy i.e. the sum of kinetic and potential energies. This equation (5) is known as *Schrödinger's wave equation* for non-relativistic motion of the particle. This equation has finite solutions only for special values of the energy E. For each such value of E, the system is said to be in corresponding *energy state*. These values of E are called *energy levels*.

We can write the Schrödinger's wave equation (5) in alternative forms. In this case the Hamiltonian of the particle can be written as

$$H(q,p) = \frac{1}{2m}\left[p_x^2 + p_y^2 + p_z^2\right] + V(x,y,z).$$

Hence the Schrödinger's wave equation becomes

$$H\left(q, -i\hbar \frac{\partial}{\partial q}\right)\psi = E\psi.$$

Since we know that

$$E\psi = i\hbar \frac{\partial \psi}{\partial t},$$

this equation becomes

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi. \tag{6}$$

This function ψ i.e. the solution of wave equation is known as *wave function* [1,3,10,12,14,15,17].

II. THE RELATIVISTIC WAVE EQUATION OF ELECTRON

The *principle of relativity* states that the laws of physics are the same for two observers moving with constant velocity relative to each other. A physical law is expressed by a mathematical equation. This equation should be of the same form in all the *Lorentz frames*, i.e. it should be invariant under the Lorentz transformation. An *electron* is an elementary particle i.e. a quanta of electromagnetic field having a *spin* of magnitude $\frac{1}{2}\hbar$. We know that the relativistic Hamiltonian H for the free particle with rest mass m is given by the formula

$$H = c(m^2c^2 + p_x^2 + p_y^2 + p_z^2)^{\frac{1}{2}} . \qquad (7)$$

In quantum mechanical treatment, H is represented by a linear operator and the components of the momentum are represented by the operators,

$$p_x = -i\hbar\frac{\partial}{\partial x}, \quad p_y = -i\hbar\frac{\partial}{\partial y}, \quad p_z = -i\hbar\frac{\partial}{\partial z}.$$

The Schrödinger wave equation obtained in (6) is

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi ,$$

with the Hamiltonian (7) equation becomes

$$i\hbar\frac{\partial}{\partial t}\psi = c(m^2c^2 + p_x^2 + p_y^2 + p_z^2)^{\frac{1}{2}}\psi$$

$$i\hbar\frac{\partial}{\partial (ct)}\psi = \left\{m^2c^2 + \left(-i\hbar\frac{\partial}{\partial x}\right)^2 + \left(-i\hbar\frac{\partial}{\partial y}\right)^2 + \left(-i\hbar\frac{\partial}{\partial z}\right)^2\right\}^{\frac{1}{2}}\psi$$

$$i\hbar\frac{\partial}{\partial x^0}\psi = \left\{m^2c^2 + \left(-i\hbar\frac{\partial}{\partial x^1}\right)^2 + \left(-i\hbar\frac{\partial}{\partial x^2}\right)^2 + \left(-i\hbar\frac{\partial}{\partial x^3}\right)^2\right\}^{\frac{1}{2}}\psi,$$

where we have written $ct = x^0$, $x = x^1$, $y = x^2$ and $z = x^3$. Now writing

$$i\hbar\frac{\partial}{\partial x^0} = p_0, \quad i\hbar\frac{\partial}{\partial x^1} = p_1, \quad i\hbar\frac{\partial}{\partial x^2} = p_2, \quad i\hbar\frac{\partial}{\partial x^3} = p_3,$$

we get

$$\left\{p_0 - (m^2c^2 + {p_1}^2 + {p_2}^2 + {p_3}^2)^{\frac{1}{2}}\right\}\psi = 0. \qquad (8)$$

This wave equation as stated by Prof. P.A.M. Dirac [3], [4] is unsatisfactory from the point of view of relativistic theory and he obtained that the equation

$$\{p_0 - \alpha_1 p_1 - \alpha_2 p_2 - \alpha_3 p_3 - \beta\}\psi = 0, \qquad (9)$$

is the desired wave equation. In equation (9) ψ stands for a vector and the quantity before ψ is a linear operator i.e. α_1, α_2, α_3, β, p_0, p_1, p_2, p_3 are also linear operators. p_0, p_1, p_2, p_3 are defined as above and α_1, α_2, α_3, β are independent of p's. They are such that if $\beta = \alpha_m mc$ then

$$\alpha_a\alpha_b + \alpha_b\alpha_a = 2\delta_{ab} \qquad (a, b = 1,2,3, or\ m), \qquad (10)$$

here δ_{ab} is the well known *Kronecker delta function* i.e.

$$\delta_{ab} = 0, \quad \text{if } a \neq b \quad \text{and} \quad \delta_{ab} = 1, \quad \text{if } a = b.$$

Thus four α's all anticommute with one another and the square of each is identity operator.

We should note here that we are adding and multiplying the operators and getting an operator i.e. we are concern here with the algebra of operators. We know that if V is a vector space over a field F, then the set A(V) of linear operators on V forms an *algebra* [7], [8], [16] with unit element. We should also have in mind that A(V) plays the universal role in the sense that every algebra with unit element over F is *isomorphic* to a *subalgebra* of A(V) for some vector space V over F [7].

Since our operator α_a $(a = 1,2,3, m)$ are real in the sense that they have real eigenvalues 1 and -1 because ${\alpha_a}^2 = 1$. Hence if we represent concerning algebra by selecting V(F)=C^4(C) i.e. represent α_a by operators on C^4 then following Dirac [3], [4] we get

$$\alpha_1 = \rho_1\sigma_1, \alpha_2 = \rho_1\sigma_2, \alpha_3 = \rho_1\sigma_3, \alpha_m = \rho_3. \qquad (11)$$

Here

$$\sigma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix},$$

$$\sigma_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\rho_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \rho_2 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix},$$

$$\rho_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

We may verify that the σ's and ρ's are all *Hermitian*, hence α's are also Hermitian. The Hermitian matrices have *real eigenvalues* i.e. real operators correspond to the Hermitian matrices.

The wave equation (8) can be written as

$$\{p_0 - \rho_1(\bar{\sigma}, \bar{p}) - \rho_3 mc\}\psi = 0, \qquad (12)$$

where $\bar{\sigma} = \sigma_1\hat{\imath} + \sigma_2\hat{\jmath} + \sigma_3\hat{k}$ and $\bar{p} = p_1\hat{\imath} + p_2\hat{\jmath} + p_3\hat{k}$.

This is field free wave equation for the electron [1], [10], [12], [14], [15]. If an *electromagnetic field* is present, then the required wave equation is

$$\left\{p_0 + \frac{e}{c}A_0 - \rho_1\left(\bar{\sigma}, \bar{p} + \frac{e}{c}\bar{A}\right) - \rho_3 mc\right\}\psi = 0, \quad (13)$$

where A_0 is *scalar potential* and $\bar{A}$ is *vector potential* of the field.

To prove that wave equation (13) is relativistic, we require to show that it is *invariant* under the *infinitesimal Lorentz transformation*.

If p_μ is a 4-vector in one Lorentz frame and p_μ^* is a 4-vector in another Lorentz frame such that

$$p_\mu^* = p_\mu + a_\mu{}^\nu p_\nu\,, \qquad (\mu, \nu = 0,1,2,3)$$

where the $a_\mu{}^\nu$ are small numbers of the first order, then it is required that

$$p_\mu^* p^{\mu *} = p_\mu p^\mu.$$

Such transformation from p_μ to $p_\mu{}^*$ is called an infinitesimal Lorentz transformation. We can see that this condition is satisfied if the quantities $a_\mu{}^\nu$ are *antisymmetric* i.e.

$$a_\mu{}^\nu + a_\nu{}^\mu = 0\,,$$

or by raising the indices by fundamental metric tensor $g^{\mu\nu}$, we can write $a^{\mu\nu} + a^{\nu\mu} = 0$.

It has been proved by Dirac [3] that the wave equation (13) is invariant under such transformation.

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